THE BASIC BUNDLE GERBE ON UNITARY GROUPS

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Abstract. We consider the construction of the basic bundle gerbe on $SU(n)$ introduced by Meinrenken and show that it extends to a range of groups with unitary actions on a Hilbert space including $U(n)$, $\mathbb{T}^n$ and $U_p(H)$, the Banach Lie group of unitaries differing from the identity by an element of a Schatten ideal. In all these cases we give an explicit connection and curving on the basic bundle gerbe and calculate the real Dixmier-Douady class. Extensive use is made of the holomorphic functional calculus for operators on a Hilbert space.

Contents

1. Introduction

Gerbes were introduced by Giraud [16] to study non-abelian cohomology. Brylinski popularised them in his book [4], in particular the case of interest here, which is gerbes with band the sheaf of smooth $U(1)$ valued functions. To every such gerbe is associated a characteristic class in $H^3(M,\mathbb{Z})$, the Dixmier-Douady class of the gerbe. Equivalence classes of gerbes on M are, through this characteristic class, in bijective correspondence with $H^3(M,\mathbb{Z})$. Therefore it is natural to look for gerbes on manifolds with non-zero degree three integer cohomology and one important example of such a manifold is a compact, simple and simply connected Lie group.

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Recall that if G is a compact, simple and simply connected Lie group then $H^3(G,\mathbb{Z}) = \mathbb{Z}$ and there is a canonical closed three-form ν on G — the *basic three-form* (see for instance [23]). ν is a de Rham representative for the image in real cohomology of the generator of $H^3(G,\mathbb{Z})$. The three-form ν is given by

$$\nu = \frac{1}{12}\langle\theta_L, [\theta_L, \theta_L]\rangle$$

where θ_L is the left Maurer-Cartan 1-form on G and $\langle\, , \,\rangle$ is the *basic* inner product on $\mathfrak{g}$ [23]. In the case where $G = SU(n)$ the basic three-form is

$$-\frac{1}{24\pi^2}\,\mathrm{tr}(g^{-1}dg)^3. \tag{1.1}$$

Although the unitary group $U(n)$ is not simply connected, we still have the isomorphism $H^3(U(n),\mathbb{Z}) = \mathbb{Z}$. The three-form (1.1) on $SU(n)$ is clearly the restriction of a closed 3-form defined on $U(n)$. This three-form is the image in real cohomology of the generator of $H^3(U(n),\mathbb{Z})$ — we will refer to it as the basic three-form on $U(n)$.

The basic three-form ν was exploited to good effect in Witten's paper [28] on WZW models. Witten considered a non-linear sigma-model in which the fields of the theory were smooth maps $g\colon \Sigma \to G$ from a compact Riemann surface Σ to a compact, simple and simply connected Lie group G. In constructing a conformally invariant action for this sigma-model Witten was lead to consider the Wess-Zumino term

$$S_{WZ}(g) = \int_B \tilde{g}^*\nu.$$

Here B is a 3-manifold with boundary Σ and $\tilde{g}\colon B \to G$ is an extension of g to B. The question arises as to whether the Wess-Zumino term is well defined. It turns out that under these topological assumptions on G, one can always find such an extension of g, and, due to the integrality property of the basic three-form ν, $\exp(2\pi i S_{WZ}(g))$ is well defined. It is natural to wonder if it is possible to remove the topological assumptions on G and make sense of this action when G is a non-simply connected group. The theory of gerbes provides a valuable way of thinking about this problem (see for example [25]). One can interpret the action $\exp(2\pi i S_{WZ}(g))$ as the holonomy over Σ of a canonically defined gerbe on G — the *basic gerbe*. This basic gerbe on G can be constructed even when the group G is not simply connected. Since the holonomy of a gerbe on a manifold can be defined irrespective of whether the manifold is simply connected or not, we see that by *defining* the action to be the holonomy of the basic gerbe over Σ, we can remove this topological assumption on the group G.

There have been a number of constructions of gerbes and bundle gerbes on a Lie group G in the literature since Brylinksi's book [4] appeared and we review them briefly to put the results of this paper into perspective. Indeed the first such construction appeared in [4] (and later in [5]); it involved the path-fibration of G and thus was inherently infinite-dimensional. It was pointed out in [4] that it would be interesting to have a finite dimensional construction.

The notion of bundle gerbe was introduced by the first author in [19]. The relationship of bundle gerbes with gerbes is analogous to that between line bundles thought of as fibrations and line bundles thought of as locally free sheaves of modules. Bundles gerbes correspond to fibrations of groupoids whereas gerbes involve sheaves of groupoids. In [19] the tautological bundle gerbe was introduced. This

was a bundle gerbe associated to any integral, closed three-form on a 2-connected manifold M. This implicitly includes the case of compact, simple, simply-connected Lie groups which was discussed more explicitly in [11]. Again these constructions are infinite-dimensional and related to the path fibration. There is a simple way of defining this bundle gerbe using the so-called lifting bundle gerbe described in [19]. The path-fibration over G is a principal bundle with structure group ΩG, the group of based loops in G, and there is a well known central extension $\widehat{\Omega G}$ of ΩG by the circle (see [23]) which one can use to form a bundle gerbe. This bundle gerbe measures the obstruction to lifting the path-fibration to a bundle with structure group $\widehat{\Omega G}$.

The next construction, due to Brylinski [6] (see also [7]), involves the Weyl map

$$\begin{array}{ccc} G/T \times T & \to & G \\ (gT, t) & \mapsto & gtg^{-1} \end{array}$$

A gerbe was defined on $G/T \times T$ using a 'cup product' construction involving line bundles on G/T and functions on T. It was shown, using some delicate sheaf arguments, that this gerbe pushes forward via the Weyl map to a gerbe on G. Brylinski notes that this construction gives an equivariant gerbe for the conjugation action of G on G. This construction of Brylinski seems to be the most general construction to date, however the geometry of this gerbe has not been explored in full detail.

Following this construction of Brylinski's was a construction of Gawedzki and Reis [14] for the case when $G = SU(n)$. The case of quotients of $SU(n)$ by finite subgroups of the centre was also treated. The methods used in this construction involved ideas from representation theory. Gawedzki and Reis also defined a connection and curving on their bundle gerbe. This work was followed shortly afterwards by a paper of Meinrenken [18]. This gave a definitive treatment of the case where G was an arbitrary compact, simple and simply connected Lie group. This construction was also representation theoretic in nature and involved the structure of the sets of regular and singular elements of G. Meinrenken's paper also gave an extensive discussion of equivariant bundle gerbes; the basic bundle gerbe constructed in the paper was shown to be equivariant and equipped with an equivariant connection and curving. A simpler construction of a local bundle gerbe in the sense of Chatterjee-Hitchin was also given for the case of $G = SU(n)$ — we shall comment more on this below.

Equivariant gerbes were also studied in the paper [3] of Behrend, Xu and Zhang; the authors constructed a bundle gerbe using the path-fibration and equipped it with an equivariant connection and curving. This construction was followed shortly by a paper of Gawedski and Reis [15] giving a generalisation of Meinrenken's construction to the case of non-simply connected groups.

Finally we come to the case of interest in this paper which is the construction of a local bundle gerbe over $SU(n)$. As mentioned above this example first appeared in the paper [18] of Meinrenken and was later discussed also by Mickelsson [21]. In this construction a local bundle gerbe was defined over the open cover of $SU(n)$ by open sets U_z consisting of unitary matrices for which z is not an eigenvalue.

We will show how to remove the dependence on the local cover in Meinrenken's and Mickelsson's construction and to generalise it to any group G which is one

of the following: the unitary group $U(n)$ (more generally the group $U(H)$ of unitary operators on some finite dimensional Hilbert space H), the (diagonal) torus $T = \mathbb{T}^n \subset U(n)$, or one of the Banach Lie groups $U_p(H)$ for H an infinite dimensional separable complex Hilbert space. These are defined in more detail below in Section 3. A common feature of all these groups is that they have natural unitary representations on finite or infinite dimensional Hilbert spaces and so, for convenience, we refer to them as unitary groups. If $G = U(n)$ or $G = U_p(H)$ then $H^3(G, \mathbb{Z}) = \mathbb{Z}$ and the image in real cohomology of the generator of $H^3(G, \mathbb{Z})$ is represented by the basic three-form (1.1) (note the trace in (1.1) makes sense for $g \in U_p(H)$ if $1 \leq p \leq 2$). Our main result is an explicit construction of a natural connection and curving on this bundle gerbe which simultaneously covers all of the above examples of unitary groups. Of particular interest is the fundamental role played by the holomorphic functional calculus for operators which allows us to obtain explicit, albeit slightly complicated, formulae for the curving (note that the functional calculus was also used in [21] to construct trivialisations of a certain gerbe). We show explicitly that the three-curvature of this connection and curving is $2\pi i$ times the basic three-form on G and hence that the basic three-form represents the real Dixmier-Doudy class of the basic bundle gerbe. The construction of the basic bundle gerbe that we give is manifestly equivariant but we do not attempt to construct an equivariant connection and curving.

We should also comment on the relation of the basic gerbe to the 'index gerbe' considered in a number of papers, beginning with [10] and further studied in [8, 17]. Carey and Wang in [12] gave a simpler treatment, which clarified the relationship of this index gerbe to the families index theorem. In [21] (see also [9]) it is explained how the basic gerbe on G can be regarded as a special case of the index gerbe: one regards points in G as specifying holonomies of connections on the trivial G-bundle over S^1; these connections can then be coupled to the Dirac operator $-i\frac{d}{dz}$ on S^1, giving a family of self adjoint operators parametrised by G. In the above cited works on the index gerbe it is evident that having explicit formulae for connections and curving is of some importance.

Finally, in summary, the first two sections give necessary background on bundle gerbes and holomorphic functional calculus. The so-called basic bundle gerbe is introduced in the next section and this is followed by a construction of a connection and curving on it. The Weyl map is used to prove that the basic three-form represents the real Dixmier-Douady class of the basic bundle gerbe. In the final section we comment briefly upon the equivariant case. A number of complicated proofs using the holomorphic functional calculus have been relegated to the appendices.

2. Background material

2.1. **Bundle gerbes.** We review briefly the definition of bundle gerbes [19]. It will be convenient to use hermitian line bundles in the definition instead of $U(1)$ principal bundles. These two approaches are, of course, equivalent. Let $\pi \colon Y \to M$ be a surjective submersion and let $Y^{[p]}$ be the p-fold fibre product

$$Y^{[p]} = \{(y_1, \dots, y_p) \mid \pi(y_1) = \cdots = \pi(y_p)\} \subset Y^p.$$

For each $i = 1, \dots, p+1$ define the projection $\pi_i \colon Y^{[p+1]} \to Y^{[p]}$ to be the map that omits the i-th element. If $J \to Y^{[p]}$ is a line bundle we define a new line bundle

$\delta(J) \to Y^{[p+1]}$ by

$$\delta(J) = \pi_1^*(J) \otimes \pi_2^*(J)^* \otimes \pi_3^*(J) \otimes \cdots .$$

It is straightforward to check that $\delta(\delta(J))$ is canonically trivial. Moreover if σ is a section of J then $\delta(\delta(\sigma)) = 1$ under this canonical trivialisation. We then have:

Definition 2.1. A *bundle gerbe* on M is a pair (L, Y) where $\pi\colon Y \to M$ is a surjective submersion and $L \to Y^{[2]}$ a line bundle such that:

(1) there is a non-zero section s of $\delta(L) \to Y^{[3]}$, and
(2) $\delta(s) = 1$ as a section of $\delta(\delta(L))$.

If L is a hermitian line bundle and s has length one we call (L, Y) a *hermitian bundle gerbe* on M. Notice that if $(y_1, y_2, y_3) \in Y^{[3]}$ then a vector

$$s(y_1, y_2, y_3) \in L_{(y_2,y_3)} \otimes L^*_{(y_1,y_3)} \otimes L_{(y_1,y_2)}$$

of length one defines a hermitian isomorphism

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \to L_{(y_1,y_3)} \tag{2.1}$$

called the *bundle gerbe multiplication.* If $(y_1, y_2, y_3, y_4) \in Y^{[4]}$ there are two natural ways to compose the bundle gerbe multiplication and define an isomorphism:

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \otimes L_{(y_3,y_4)} \to L_{(y_1,y_4)}.$$

If these two isomorphisms agree we call the bundle gerbe multiplication *associative* and this is equivalent to the condition $\delta(s) = 1$.

Associated to any bundle gerbe is a class in $H^3(M, \mathbb{Z})$ called the *Dixmier-Douady class* of the bundle gerbe. We will omit its derivation, which can be found in [19], as we do need it in the discussion below. We do, however, need to understand its image in real cohomology which can be defined as follows.

Let $\Omega^q(Y^{[p]})$ be the space of differential q-forms on $Y^{[p]}$. Define a homomorphism

$$\delta\colon \Omega^q(Y^{[p]}) \to \Omega^q(Y^{[p+1]}) \quad \text{by} \quad \delta = \sum_{i=1}^{p+1} (-1)^{i-1} \pi_i^*.$$

These maps form the *fundamental complex*

$$0 \to \Omega^q(M) \xrightarrow{\pi^*} \Omega^q(Y) \xrightarrow{\delta} \Omega^q(Y^{[2]}) \xrightarrow{\delta} \Omega^q(Y^{[3]}) \xrightarrow{\delta} \dots$$

which is exact [19]. If (L, Y) is a bundle gerbe on M then a *bundle gerbe connection* is a connection ∇ on L such that s is covariantly constant for $\delta(\nabla)$. Equivalently the connection ∇ commutes with the bundle gerbe multiplication (2.1). If ∇ is a bundle gerbe connection and F_∇ is its curvature then $\delta(F_\nabla) = 0$ so that $F_\nabla = \delta(f)$ for some two-form $f \in \Omega^2(Y)$. A choice of such an f is called a *curving* for ∇. From the exactness of the fundamental complex we see that the curving is only unique up to addition of two-forms pulled back to Y from M. Given a choice of curving f we have $\delta(df) = d\delta(f) = dF_\nabla = 0$ so that $df = \pi^*(\omega)$ for a closed three-form ω on M called the *three-curvature* of ∇ and f. The de Rham class

$$\left[\frac{1}{2\pi i}\omega\right] \in H^3(M, \mathbb{R})$$

is an integral class which is the image in real cohomology of the Dixmier-Douady class of (L, Y). For convenience let us call this the real Dixmier-Douady class of (L, Y).

2.2. **Holomorphic functional calculus.** We briefly recall the main features of the holomorphic functional calculus and refer the reader to [13] for more details. Let $B(H)$ denote the Banach algebra of bounded operators on a Hilbert space H. Suppose that $T \in B(H)$ and that the spectrum $\mathrm{spec}(T)$ of T is a compact subset of the complex plane. Given a complex valued function f, holomorphic on an open neighbourhood U of $\mathrm{spec}(T)$, we can define a new operator $f(T)$ by the contour integral

$$f(T) = \frac{1}{2\pi i} \oint_C f(\xi)(\xi - T)^{-1} d\xi.$$

Here C is a contour surrounding $\mathrm{spec}(T)$ which lies entirely in U and is always taken to be oriented counter-clockwise. Because the resolvent $(\xi - T)^{-1}$ is holomorphic away from $\mathrm{spec}(T)$ this definition is independent of continuous deformations of C. It is an easy consequence of Cauchy's theorem that if g is another complex valued function, holomorphic on the same open neighbourhood U, then $(fg)(T) = f(T)g(T)$.

As an example of this formula suppose that λ is an isolated point of $\mathrm{spec}(T)$ which is an eigenvalue of T. Then the orthogonal projection P_λ onto the λ eigenspace of T, $E_{(T,\lambda)}$, is given by the contour integral

$$P = \frac{1}{2\pi i} \oint_{C_\lambda} (\xi - T)^{-1} d\xi \tag{2.2}$$

where C_λ is a small circle centred at λ which contains no other point of $\mathrm{spec}(T)$. It is instructive to consider this for the case when $T = g$ is a unitary matrix. Since we can write g as a sum $\sum \lambda_i P_i$ where P_i denotes the orthogonal projection onto the λ_i-eigenspace we see that the resolvent $(\xi - g)^{-1}$ can be written as a sum

$$(\xi - g)^{-1} = \sum (\xi - \lambda_i)^{-1} P_i.$$

This gives an effective way to evaluate the contour integral (2.2) since this can now be written as

$$\frac{1}{2\pi i} \oint_{C_\lambda} \sum (\xi - \lambda_i)^{-1} P_i d\xi$$

and all one has to do is evaluate the contour integrals $\frac{1}{2\pi i} \oint_{C_\lambda} (\xi - \lambda_i)^{-1} d\xi$ which can easily be done.

In a similar fashion assume that C is a contour encircling eigenvalues $\lambda_1, \ldots, \lambda_r$ of T. Then

$$P = \frac{1}{2\pi i} \oint_C (\xi - T)^{-1} d\xi \tag{2.3}$$

gives the projection onto the sum of the eigenspaces:

$$E_{(T,\lambda_1)} \oplus \cdots \oplus E_{(T,\lambda_r)}$$

3. The basic bundle gerbe

We now give a global version of the construction in [18, 21] of a bundle gerbe on a group of unitary operators. Suppose that G is one of the following groups: the unitary group $U(n)$ (more generally the group $U(H)$ of unitary operators on some finite dimensional Hilbert space H), the (diagonal) torus $T = \mathbb{T}^n \subset U(n)$, or one of the Banach Lie groups $U_p(H)$ for H an infinite dimensional separable complex Hilbert space. Recall [24] that for $p \geq 1$ the groups $U_p(H)$ are defined to be the groups of unitary operators on H differing from the identity by an operator in the

Schatten ideal $\mathcal{L}_p(H)$. For more details on the ideals $\mathcal{L}_p(H)$ we refer the reader to [26]. Notice that in all cases elements of G act as unitary operators on a Hilbert space H. For convenience of presentation we shall refer to the space that G acts on as H even when it might be more natural to call it $\mathbb{C}^n$.

Denote the identity in G by 1 and note that if $g \in G$ then its spectrum $\operatorname{spec}(g)$ (when we consider g as an operator on H) is a subset of the circle $U(1) \subset \mathbb{C}$. If H is finite dimensional then the spectrum of $g \in G$ is of course finite, and hence discrete. Suppose that H is infinite dimensional so that $g \in U_p(H)$ for some $p \geq 1$. Write $g = 1 + A$ where A belongs to the appropriate ideal $\mathcal{L}_p(H)$. If $\lambda \in \operatorname{spec}(g)$ then $g - \lambda$ is not invertible and hence $A - (\lambda - 1)$ is not invertible. This means $\lambda - 1$ belongs to the spectrum of the compact operator A. The spectrum $\operatorname{spec}(A)$ of the compact operator A is a set with no non-zero accumulation points, and if $\mu \in \operatorname{spec}(A)$ is non-zero then μ is an eigenvalue of finite multiplicity. It follows that the spectrum $\operatorname{spec}(g)$ of g is a subset of $U(1)$ which has at most one accumulation point at $1 \in U(1)$. All elements of the spectrum of g not equal to 1 are eigenvalues of finite multiplicity. Denote by $U_0(1)$ the set $U(1)$ with the identity element, 1, removed.

Define
$$Y = \{(z, g) \mid z \notin \operatorname{spec}(g) \cup \{1\}\} \subset U_0(1) \times G$$
and let $\pi\colon Y \to G$ be the projection $\pi(z, g) = g$. We will identify $Y^{[p]}$ with the subset of $U_0(1)^p \times G$ determined by
$$Y^{[p]} = \{(z_1, \ldots, z_p, g) \mid z_1, \ldots, z_p \notin \operatorname{spec}(g) \cup \{1\}\} \subset U_0(1)^p \times G$$
We put an ordering on $U_0(1)$ as follows. If $z_1, z_2 \in U_0(1)$ we say that $z_1 > z_2$ if we can positively rotate z_2 into z_1 without passing through 1. For any pair $z_1, z_2 \in U_0(1)$ we say that λ is *between* z_1 and z_2 if λ is in the connected component of $U(1) - \{z_1, z_2\}$ not containing 1. Call a point $(z_1, z_2, g) \in Y^{[2]}$ *positive* if there are eigenvalues of g between z_1 and z_2 and $z_1 > z_2$, *null* if there are no eigenvalues of g between z_1 and z_2 and *negative* if there are eigenvalues of g between z_1 and z_2 and $z_1 < z_2$. Note that if (z_1, z_2, g) is positive (negative) then (z_2, z_1, g) is negative (positive). Denote the corresponding subsets of $Y^{[2]}$ by $Y^{[2]}_+$, $Y^{[2]}_0$ and $Y^{[2]}_-$.

Let $E_{(g,\lambda)}$ be the λ eigenspace of g for $\lambda \in U_0(1)$. If (z_1, z_2, g) is in $Y^{[2]}_+$ we define
$$E_{(z_1,z_2,g)} = \bigoplus_{z_1 > \lambda > z_2} E_{(g,\lambda)}. \tag{3.1}$$
By construction, $E_{(z_1,z_2,g)}$ has finite dimension so we can define the top exterior power
$$L_{(z_1,z_2,g)} = \det(E_{(z_1,z_2,g)})$$
If $(z_1, z_2, g) \in Y^{[2]}_0$ we define
$$L_{(z_1,z_2,g)} = \mathbb{C}$$
and if $(z_1, z_2, g) \in Y^{[2]}_-$ we define
$$L_{(z_1,z_2,g)} = \det(E_{(z_2,z_1,g)})^*.$$

We want to show that $L \to Y^{[2]}$ is a smooth locally trivial hermitian line bundle. It suffices to show that $E \to Y^{[2]}_+$ is smooth and locally trivial and by standard results this follows if we can show that $P\colon Y^{[2]}_+ \to B(H)$ is smooth where P is the

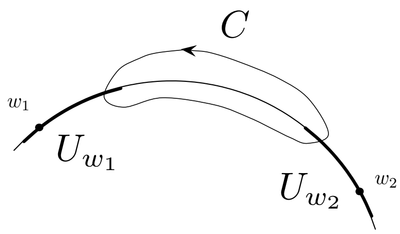

FIGURE 3.1. The contour C.

orthogonal projection onto E. To show this last fact we will use the holomorphic functional calculus from Section 2.2.

Consider the continuous map

$$\begin{array}{ccc} U_0(1) \times U_0(1) \times G & \to & B(H) \times B(H) \\ (z_1, z_2, g) & \mapsto & (z_1 - g, z_2 - g) \end{array} \tag{3.2}$$

As $Y^{[2]}$ is the pre-image under this of the open set of pairs of invertible operators in $B(H) \times B(H)$ it follows that $Y^{[2]}$ is open in $U_0(1) \times U_0(1) \times G$. If $(w_1, w_2, h) \in Y_+^{[2]}$ we can therefore find connected open sets U_{w_1}, U_{w_2} and U_h containing w_1, w_2 and h respectively, with the property that if

$$(z_1, z_2, g) \in U_{w_1} \times U_{w_2} \times U_h \subset Y^{[2]}$$

then neither of z_1 or z_2 is in the spectrum of g. In particular, if $g \in U_h$ and C is any anti-clockwise contour cutting $U(1)$ once in U_{w_1} and once in U_{w_2} then C must encircle all the eigenvalues of g between z_1 and z_2. Fix such a contour as in Figure 3.1. Let

$$U_{(w_1, w_2, h)} = U_{w_1} \times U_{w_2} \times U_h. \tag{3.3}$$

The projection map restricted to $U_{(w_1, w_2, h)}$

$$P \colon U_{(w_1, w_2, h)} \to B(H)$$

is therefore given by the contour integral formula

$$P(z_1, z_2, g) = \frac{1}{2\pi i} \oint_C (\xi - g)^{-1} d\xi. \tag{3.4}$$

As we can differentiate through the contour integral and the integrand is smooth it follows that P is smooth on $U_{(w_1, w_2, h)}$ and hence on all of $Y^{[2]}$. Moreover $\mathrm{tr}(P) = \dim(E_{(z_1, z_2, g)})$ must be constant and equal to $\dim(E_{(w_1, w_2, h)})$ so that $U_{(w_1, w_2, h)}$ must lie entirely in one of $Y_+^{[2]}$, $Y_0^{[2]}$ or $Y_-^{[2]}$ and hence they must all be open. It follows that $P \colon Y_+^{[2]} \to B(H)$ is smooth and we have shown that:

Proposition 3.1. *L is a smooth and locally trivial line bundle on $Y^{[2]}$.*

We want to now prove that L is a bundle gerbe. The basic fact that this uses is that if $V \oplus W$ is a direct sum of vector spaces then wedge product defines a canonical isomorphism

$$\det(V) \otimes \det(W) = \det(V \oplus W)$$

However the proof is made complicated by the various cases that arise. To handle some of these we make the following definitions. Let Σ_r denote the symmetric group on r letters. This acts naturally on $Y^{[r]}$ by permuting the $y_1, \ldots, y_r$ in $(y_1, \ldots, y_r, g)$. If $J \to Y^{[r]}$ is a line bundle we say that J is *antisymmetric* if for all $\rho \in \Sigma_r$ we have isomorphisms

$$J \simeq \rho^*(J)^{\mathrm{sgn}(\rho)}$$

where we denote by $\rho\colon Y^{[r]} \to Y^{[r]}$ the map induced by the permutation ρ. Also if V is a vector space, we are using the notation V^{-1} for V^*. If $0 \neq v \in V$ a one-dimensional vector space define $v^* \in V^*$ by $v(v^*) = 1$ and also denote v^* by v^{-1}. If J is antisymmetric and ψ is a non-vanishing section of J then it makes sense to define ψ to be antisymmetric if $\psi = \rho^*(\psi)^{\mathrm{sgn}(\rho)}$ for all $\rho \in \Sigma_r$.

Lemma 3.2. *If $J \to Y^{[r]}$ is antisymmetric then $\delta(J)$ is antisymmetric. In such a case if ψ is an antisymmetric section of J then $\delta(\psi)$ is an antisymmetric section of $\delta(J)$.*

Proof. We have

$$\delta(J)_{(x_1,x_2,\ldots,x_{r+1})} = J_{(x_2,x_3,\ldots,x_{r+1})} \otimes J^*_{(x_1,x_3,\ldots,x_{r+1})} \otimes J_{(x_1,x_2,x_4,\ldots\ldots,x_{r+1})} \otimes \cdots$$

Inspection shows that any side by side transposition changes the right hand side to its dual so the result follows. The same method works for ψ. □

Notice that, by construction, $L \to Y^{[2]}$ is an antisymmetric hermitian line bundle. Consider $(z_1, z_2, z_3, g) \in Y^{[3]}$. There is a $\rho \in \Sigma_3$ such that $z_{\rho(1)} \geq z_{\rho(2)} \geq z_{\rho(3)}$. Define (z_1, z_2, z_3, g) to be of type $(1,1)$ if there are eigenvalues of g between $z_{\rho(1)}$ and $z_{\rho(2)}$ and also between $z_{\rho(2)}$ and $z_{\rho(3)}$, type $(1,0)$ if there are eigenvalues of g between $z_{\rho(1)}$ and $z_{\rho(2)}$ but not between $z_{\rho(2)}$ and $z_{\rho(3)}$, type $(0,1)$ if there are no eigenvalues of g between $z_{\rho(1)}$ and $z_{\rho(2)}$ but there are some between $z_{\rho(2)}$ and $z_{\rho(3)}$ and type $(0,0)$ if there are no eigenvalues of g between $z_{\rho(1)}$ and $z_{\rho(3)}$. Notice that although ρ is not unique if some of the z_i are equal, nevertheless these definitions still make sense. Denote by $Y^{[3]}_{(i,j)}$ the subset of $Y^{[3]}$ consisting of elements of type (i,j) for each $i,j = 0,1$. Each $Y^{[3]}_{(i,j)}$ is invariant under Σ_3 and is a union of connected components of $Y^{[3]}$ and hence open.

Consider $(z_1, z_2, z_3, g) \in Y^{[3]}_{(1,1)}$ with $z_1 > z_2 > z_3$. We have

$$\left(\bigoplus_{z_1 > \lambda > z_2} E_{(g,\lambda)} \right) \oplus \left(\bigoplus_{z_2 > \lambda > z_3} E_{(g,\lambda)} \right) = \left(\bigoplus_{z_1 > \lambda > z_3} E_{(g,\lambda)} \right) \tag{3.5}$$

so that

$$E_{(z_1,z_2,g)} \oplus E_{(z_2,z_3,g)} = E_{(z_1,z_3,g)} \tag{3.6}$$

and hence wedge product gives an isomorphism

$$L_{(z_1,z_2,g)} \otimes L_{(z_2,z_3,g)} = L_{(z_1,z_3,g)}. \tag{3.7}$$

Moreover this defines a smooth section

$$s(z_1, z_2, z_3, g) \in \delta(L)_{(z_1,z_2,z_3,g)} = L_{(z_2,z_3,g)} \otimes L^*_{(z_1,z_3,g)} \otimes L_{(z_1,z_2,g)} = \mathbb{C} \tag{3.8}$$

at points satisfying $z_1 > z_2 > z_3$. We can extend this to a section s of $\delta(L)$ over all of $Y^{[3]}_{(1,1)}$ by requiring antisymmetry.

Consider $(z_1, z_2, z_3, g) \in Y^{[3]}_{(1,0)}$ with $z_1 > z_2 \geq z_3$. Then by definition

$$L_{(z_2,z_3,g)} = \mathbb{C} \tag{3.9}$$

and moreover $E_{(z_1,z_2,g)} = E_{(z_1,z_3,g)}$ so that

$$L_{(z_1,z_2,g)} = L_{(z_1,z_3,g)}. \tag{3.10}$$

Thus we have

$$\delta(L)_{(z_1,z_2,z_3,g)} = L_{(z_2,z_3,g)} \otimes L^*_{(z_1,z_3,g)} \otimes L_{(z_1,z_2,g)} = \mathbb{C} \otimes L^*_{(z_1,z_3,g)} \otimes L_{(z_1,z_3,g)} = \mathbb{C}$$

so we can define

$$s(z_1, z_2, z_3, g) \in \delta(L)_{(z_1,z_2,z_3,g)}.$$

Notice that on the connected component containing (z_1, z_2, z_3, g) the conditions in equations (3.9) and (3.10) will be satisfied for all points so that ψ extends to a smooth section on all of that connected component. A similar argument can be applied at the other points of $Y^{[3]}_{(1,0)}$ to give an antisymmetric section s. Clearly the same type of argument can be applied in the case of $Y^{[3]}_{(0,1)}$. On the remaining subset $Y^{[3]}_{(0,0)}$ all the line bundles are trivial so there is an obvious section. We conclude that there is a naturally defined antisymmetric section s of $\delta(L)$ over all of $Y^{[3]}$.

We have already remarked that $\delta(s) = 1$ is equivalent to the bundle gerbe multiplication (2.1) being associative. In the case that $z_1 > z_2 > z_3 > z_4$ and there are eigenvalues of g between each of the consecutive z's this follows because the wedge product is associative. As s and L are both antisymmetric it follows from Lemma 3.2 that $\delta(s)$ is antisymmetric, so that $\delta(s)(z_1, z_2, z_3, z_4, g) = 1$ whenever there are eigenvalues of g between each consecutive z's, regardless of their ordering. The other cases can be dealt with in a similar fashion. We conclude that $L \to Y^{[2]}$ is a hermitian bundle gerbe on G.

4. A bundle gerbe connection and its curvature

The basic bundle gerbe has a canonical bundle gerbe connection constructed as follows (this construction is mentioned in [10] and discussed also in [14]). Over $Y^{[2]}_+$ we have

$$E \subset H \times Y^{[2]}_+$$

and we can use the orthogonal projection

$$P : H \to E$$

to project the trivial connection on $H \times Y^{[2]}_+$ to a connection ∇ on E. Over $Y^{[2]}_+$ this defines a connection $\det(\nabla)$ on L, over $Y^{[2]}_-$ we take the corresponding dual connection and over $Y^{[2]}_0$ we take the flat connection. Notice that as L is antisymmetric we can consider the corresponding notion of antisymmetry for a connection and by construction $\det(\nabla)$ is antisymmetric.

We wish to show this is a bundle gerbe connection. Consider a connected component X of $Y^{[3]}_{(1,1)}$ containing some (z_1, z_2, z_3, g) for which $z_1 > z_2 > z_3$. From equation (3.6) we have the orthogonal direct sum

$$E_{(z_1,z_3,g)} = E_{(z_1,z_2,g)} \oplus E_{(z_2,z_3,g)} \subset H. \tag{4.1}$$

It follows that over X we have

$$\pi_2^*(E) = \pi_3^*(E) \oplus \pi_1^*(E) \subset Y^{[3]} \times H \tag{4.2}$$

and moreover that we have

$$\pi_2^*(\nabla) = \pi_3^*(\nabla) \oplus \pi_1^*(\nabla)$$

because equation (4.2) is an orthogonal splitting and the connections are defined by orthogonal projections. Hence over X the connection respects the wedge product isomorphism

$$\pi_2^*(L) = \pi_3^*(L) \otimes \pi_1^*(L)$$

and hence the section s is covariantly constant for the connection on L over X. By antisymmetry the section s is covariantly constant for the connection over all of $Y^{[3]}_{(1,1)}$. The other parts of $Y^{[3]}$ can be dealt with in a similar fashion. We conclude that the connection we have constructed is a bundle gerbe connection.

Over $Y^{[2]}_+$ the curvature two-form $F_{\det(\nabla)}$ of the bundle gerbe connection will be equal to the trace $\mathrm{tr}(F_\nabla)$ of the curvature F_∇ of the connection ∇ on the vector bundle E. It is a simple calculation to see that this can be computed in terms of the projections P as

$$F_{\det(\nabla)} = \mathrm{tr}(PdPdP). \tag{4.3}$$

By the antisymmetry of $\det(\nabla)$ the sign of $F_{\det(\nabla)}$ will change on $Y^{[2]}_-$ and it will vanish on $Y^{[2]}_0$.

We now explain how the holomorphic functional calculus from Section 2.2 can be used to derive a contour integral expression for the curvature two-form $F_{\det(\nabla)}$ described above. This will be useful in defining a curving for $F_{\det(\nabla)}$. We can use the local expressions (3.4) for the projections P to write down a contour integral formula for $F_{\det(\nabla)}$; a priori this will be a triple contour integral, however it is possible to obtain a simpler formula as in the following Proposition. Let G be one of the groups $U(H)$ for H a finite dimensional complex Hilbert space, or $U_p(H)$ for H an infinite dimensional complex Hilbert space and $1 \leq p \leq 2$. For any $(z_1, z_2, g) \in Y^{[2]}_+$ choose a closed contour $C_{(z_1,z_2,g)}$ enclosing all the eigenvalues of g between z_1 and z_2, oriented counter clockwise. Then we have

Proposition 4.1. *The restriction of the curvature $F_{\det(\nabla)}$ to $Y^{[2]}_+$ is given by*

$$F_{\det(\nabla)}(z_1, z_2, g) = \frac{1}{4\pi i} \oint_{C_{(z_1,z_2,g)}} \mathrm{tr}\left((\xi - g)^{-1} dg (\xi - g)^{-2} dg\right) d\xi \tag{4.4}$$

Notice that on the right hand side of this formula we have committed an abuse of notation and denoted by dg the derivative at (z_1, z_2, g) of the projection map $Y^{[2]} \to G$. Notice also that the curvature two-form has no components in the $U(1)$ directions. We also need to comment on this expression in the infinite dimensional case when $G = U_p(H)$. If $1 \leq p \leq 2$ then the trace in (4.4) makes sense, since $(\xi - g)^{-1} dg (\xi - g)^{-2} dg$ is a product of one-forms taking values in $\mathcal{L}_p(H)$, and hence is a two-form taking values in the trace class operators on H. The proof of Proposition 4.1 is deferred to Appendix A.

5. A curving

We would now like to find a curving for this bundle gerbe connection, in other words we would like to find a two-form f on Y such that

$$F_{\det(\nabla)} = \delta(f) = \pi_1^* f - \pi_2^* f$$

on $Y^{[2]}$. We can do this as follows. For each complex number z with $|z| = 1$ let R_z denote the closed ray from the origin through z. For any $z \in U_0(1)$ we define a branch of the logarithm, $\log_z \colon \mathbb{C} - R_z \to \mathbb{C}$, by making the cut along R_z and also setting $\log_z(1) = 0$. If $z_1 > z_2$ write (z_1, z_2) for the set of $\xi \in U(1)$ between z_1 and z_2 in the sense of Section 3. It is easy to check that if $z_1 > z_2$ then

$$\log_{z_1} \xi - \log_{z_2} \xi = \begin{cases} 2\pi i & \text{if } \xi/|\xi| \in (z_1, z_2) \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

For any $(z, g) \in Y$, choose an anti-clockwise closed contour $C_{(z,g)}$ in $\mathbb{C} - R_z$ which encloses $\operatorname{spec}(g)$.

Theorem 5.1. *Suppose that $G = U(H)$ for some finite dimensional complex Hilbert space H. Define a two-form f on Y by*

$$f(z, g) = \frac{1}{8\pi^2} \oint_{C_{(z,g)}} \log_z \xi \operatorname{tr}\left((\xi - g)^{-1} dg (\xi - g)^{-2} dg\right) d\xi. \tag{5.2}$$

Then we have:
(a) the two-form f is a curving for the bundle gerbe connection $\det(\nabla)$ *in the sense that it satisfies $\delta(f) = F_{\det(\nabla)}$ and,*
(b) the 3-curvature of the bundle gerbe connection $\det(\nabla)$ *and curving f is the three-form*

$$-\frac{i}{12\pi} \operatorname{tr}(g^{-1} dg)^3.$$

(c) The real Dixmier-Douady class of the basic bundle gerbe (L, Y) is

$$-\frac{1}{24\pi^2} \operatorname{tr}(g^{-1} dg)^3.$$

Again we have abused notation on the right hand side of (5.2) and denoted by dg what is really the derivative at (z, g) of the projection from $Y^{[2]}$ to G.

We need to show first of all that (5.2) defines a smooth two-form on Y. Observe that $f(z, g)$ is independent of the choice of contour $C_{(z,g)}$, since the integrand is holomorphic in ξ. Fix $(w, h) \in Y$. We can choose an open neighbourhood U_w of w in $U_0(1)$, and an open neighbourhood U_h of h in G such that $U_{(w,h)} = U_w \times U_h$ is an open neighbourhood of (w, h) in Y. After perhaps shrinking U_w a little we conclude that if $g \in U_h$ then $\operatorname{spec}(g)$ does not intersect $\overline{U}_w$. Therefore we can find a contour C which contains the spectrum of every $g \in U_h$ and lies inside

$$\mathbb{C} - \overline{\bigcup_{z \in U_w} R_z}.$$

Such a C satisfies the requirements to be a $C_{(z,g)}$ for any $(z, g) \in U_{(w,h)}$. Moreover for any $z \in U_w$ we have $\log_z = \log_w$ on C by (5.1). Hence it follows that the restriction of f to $U_{(w,h)}$ satisfies

$$f(z, g) = \frac{1}{8\pi^2} \oint_C \log_w \xi \operatorname{tr}((\xi - g)^{-1} dg (\xi - g)^{-2} dg) d\xi.$$

and is clearly smooth on this open set and hence on all of $Y_+^{[2]}$.

Part (a) of Theorem 5.1 is not too difficult to establish. It is enough to prove this on the open subset $Y_+^{[2]}$ of $Y^{[2]}$. Recall that on this set $z_1 > z_2$. Since the

expression (5.2) for $f(z_1, g)$ is independent of the choice of contour $C_{(z_1,g)}$ we can replace the contour $C_{(z_1,g)}$ with the union

$$C_{(z_1,z_2,g)} \cup \tilde{C}_{(z_1,z_2,g)}$$

where the contour $C_{(z_1,z_2,g)}$ is the one described in Proposition 4.1 above and the contour $\tilde{C}_{(z_1,z_2,g)}$ is a contour surrounding the part of the spectrum of g lying in the set $U_0(1) \setminus (z_1, z_2)$. Then we have that $\pi_2^* f$ is given by

$$\begin{aligned}
\frac{1}{8\pi^2} \oint_{C_{(z_1,g)}} & \log_{z_1} \xi \operatorname{tr}\left((\xi - g)^{-1} dg (\xi - g)^{-2} dg\right) d\xi \\
&= \frac{1}{8\pi^2} \oint_{C_{(z_1,z_2,g)}} \log_{z_1} \xi \operatorname{tr}\left((\xi - g)^{-1} dg (\xi - g)^{-2} dg\right) d\xi \\
&\qquad + \frac{1}{8\pi^2} \oint_{\tilde{C}_{(z_1,z_2,g)}} \log_{z_1} \xi \operatorname{tr}\left((\xi - g)^{-1} dg (\xi - g)^{-2} dg\right) d\xi.
\end{aligned}$$

Likewise we can write the contour $C_{(z_2,g)}$ as a union $C_{(z_1,z_2,g)} \cup \tilde{C}_{(z_1,z_2,g)}$ and obtain an expression for $\pi_1^* f$ as a sum of contour integrals as above. It then follows, using equation (5.1) that $\delta(f) = F_{\det(\nabla)}$ on $Y_+^{[2]}$.

The proof of part (b) of Theorem 5.1 requires a little preparation, in particular we need to make use of some properties of the so-called 'Weyl map'. We discuss this in the next section and complete the remainder of the proof of Theorem 5.1 in the Appendix.

It is possible to generalise Theorem 5.1 to include the Banach Lie groups $U_p(H)$ for $1 \leq p \leq 2$. More precisely we have the following result:

Theorem 5.2. *Let G be one of the Banach Lie groups $U_p(H)$ for $1 \leq p \leq 2$. Define a two-form f on Y by*

$$f(z, g) = \frac{1}{8\pi^2} \oint_{C_{(z,g)}} \log_z \xi \operatorname{tr}\left((\xi - g)^{-1} dg (\xi - g)^{-2} dg\right) d\xi. \tag{5.3}$$

Then we have:
(a) the two-form f is a curving for the bundle gerbe connection $\det(\nabla)$ *in the sense that it satisfies $\delta(f) = F_{\det(\nabla)}$ and,*
(b) the 3-curvature of the bundle gerbe connection $\det(\nabla)$ *and curving f is the three-form*

$$-\frac{i}{12\pi} \operatorname{tr}(g^{-1} dg)^3.$$

(c) The real Dixmier-Douady class of the basic bundle gerbe (L, Y) is

$$-\frac{1}{24\pi^2} \operatorname{tr}(g^{-1} dg)^3.$$

The comment about traces following Proposition 4.1 applies verbatim in this situation to show that the trace in the expression (5.3) is well-defined. The proof that f is a smooth two-form on Y and that $\delta(f) = F_{\det(\nabla)}$ goes through in exactly the same manner as above. We now need to identify the associated 3-curvature.

Let $e_1, e_2, e_3, \ldots$ be an orthonormal basis of H and let H_n be the subspace of H spanned by $e_1, e_2, \ldots, e_n$. The algebra $gl(H_n)$ of linear transformations of H_n can be identified with a subalgebra of $B(H)$ by sending a linear map T to the bounded

operator P_nTP_n (here P_n denotes the orthogonal projection onto H_n). As pointed out in [24], for any $p, n \geq 1$ there is an inclusion

$$U(H_n) \subset U_p(H),\ g \mapsto (P_n g P_n - P_n) + 1.$$

It is clear that the basic gerbe over $U_p(H)$ defined above restricts to the basic gerbe over $U(H_n)$ for any n. Also note that forms on $U_p(H)$ restrict to forms on $U(H_n)$. In [24] Quillen proves the following result:

Theorem 5.3 ([24] Proposition 7.16)**.** *Let ω be a form defined on $U_p(H)$ for any $p \geq 1$. If ω vanishes when restricted to $U(H_n)$ for any n, then ω vanishes on $U_p(H)$.*

We can use Theorem 5.3 to calculate the 3-curvature ω of the basic bundle gerbe with connection $\det(\nabla)$ and curving f on $U_p(H)$ for $1 \leq p \leq 2$. Consider the three-form

$$\omega + \frac{i}{12\pi}\operatorname{tr}(g^{-1}dg)^3$$

defined on $U_p(H)$. Under the inclusion $U(H_n) \subset U_p(H)$ the left Maurer-Cartan 1-form $\Theta_L = g^{-1}dg$ on $U_p(H)$ pulls back to a 1-form on $U(H_n)$ taking values in $\mathcal{L}_p(H)$. It is easy to see that this pullback of Θ_L is given by the 1-form

$$P_n\theta_L P_n$$

on $U(H_n)$ where θ_L denotes the left Maurer-Cartan 1-form $\theta_L = g^{-1}dg$ on $U(H_n)$. Since θ_L commutes with P_n it follows that the three-form $-\frac{i}{12\pi}\operatorname{tr}(g^{-1}dg)^3$ on $U_p(H)$ restricts to the corresponding three-form $-\frac{i}{12\pi}\operatorname{tr}(g^{-1}dg)^3$ on every $U(H_n)$ and so the difference $\omega + \frac{i}{12\pi}\operatorname{tr}(g^{-1}dg)^3$ vanishes on every $U(H_n)$. By Quillen's result this means that

$$\omega = -\frac{i}{12\pi}\operatorname{tr}(g^{-1}dg)^3.$$

6. The Weyl map $p\colon G/T \times T \to G$

Suppose that G is a compact, connected Lie group and that T is a maximal torus of G. There is a canonical map

$$p\colon G/T \times T \to G$$

defined by sending a pair (gT, t) to the element gtg^{-1} of G. Clearly this is independent of the choice of representative of the coset gT. This map has a number of very useful properties. For instance it is G-equivariant for the obvious left G action on $G/T \times T$ and the left G action by conjugation on G. More useful for us however is the following fact.

Proposition 6.1. *The map $p\colon G/T \times T \to G$ is a $|W|$ sheeted covering when restricted to the dense open subset G_{reg} of regular elements in G (here $|W|$ denotes the order of the Weyl group W of G).*

Recall (see for instance [1]) that an element g of G is said to be regular if the dimension of its centraliser is equal to $\dim T$. For $G = U(n)$ this means that g has distinct eigenvalues or that it is conjugate to a diagonal matrix $\operatorname{diag}(z_1, \dots, z_n)$ with all the z_i distinct. It is clear that the set G_{reg} of regular elements in G is a dense open subset of G.

Over G_{reg} the map p restricts to a map $p_{\text{reg}}\colon G/T \times T_{\text{reg}} \to G_{\text{reg}}$ where T_{reg} denotes the dense open subset of T consisting of regular elements. As mentioned in

the statement of Proposition 6.1 the map p_{reg} is a $|W|$ sheeted covering. This means in particular the derivative dp_{reg} is surjective and so the pullback map $(p_{\mathrm{reg}})^*\colon \Omega(G_{\mathrm{reg}}) \to \Omega(G/T \times T_{\mathrm{reg}})$ is injective on forms. However since G_{reg} is dense in G it follows that the pullback map $p^*\colon \Omega(G) \to \Omega(G/T \times T)$ must also be injective. We record the above discussion in the following Corollary to Proposition 6.1:

Corollary 6.2. *The Weyl map $P\colon G/T \times T \to G$ induces by pullback an injective map on differential forms*

$$p^*\colon \Omega(G) \to \Omega(G/T \times T).$$

For the special case when $G = U(n)$ and T is the diagonal torus the map p has a nice description. First of all G/T can be identified with the flag manifold F_n of $\mathbb{C}^n$, so that we can think of a point in G/T as an increasing sequence

$$V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$$

of subspaces of $\mathbb{C}^n$ such that $\dim V_{i+1}/V_i = 1$, or alternatively as a family of 1-dimensional subspaces $W_1, W_2, \ldots, W_n$ of $\mathbb{C}^n$ such that W_i is orthogonal to W_j if $i \neq j$. Replacing the subspaces W_i with the orthogonal projections P_i onto them, we see that we can identify a point in G/T with a family of orthogonal projections $P_1, P_2, \ldots, P_n$ such that $P_iP_j = 0$ if $i \neq j$ and $\sum_i P_i = 1$.

So we can think of a point in $G/T \times T$ as a pair (P, λ) consisting of a family $P = (P_1, P_2, \ldots, P_n)$ of such orthogonal projections and a point $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ in T. It is useful to regard the λ_i as the eigenvalues of a unitary matrix g and the P_i as the orthogonal projections onto the corresponding eigenspaces. With this interpretation the map $p\colon G/T \times T \to G$ has a simple description: it is the map which sends such a pair (P, λ) to the unitary matrix

$$g = \sum_{i=1}^{n} \lambda_i P_i.$$

As we have already mentioned, the construction of the basic bundle gerbe in Section 3 gives in particular a bundle gerbe over the (diagonal) maximal torus $T = \mathbb{T}^n$ of $G = U(n)$. Let us denote by (L_T, Y_T) this basic bundle gerbe on T so that Y_T is the subset of $U(1)_0 \times T$ consisting of pairs (z, t) so that z is not one of the diagonal entries of t. Note that the map $p\colon G/T \times T \to G$ induces a map

$$\begin{aligned} p_Y\colon G/T \times Y_T &\to Y \\ (gT, (z, t)) &\mapsto (z, gtg^{-1}). \end{aligned}$$

Let Y_{reg} be the subset of Y consisting of pairs (z, g) where $g \in G_{\mathrm{reg}}$. Similarly let $Y_{T,\mathrm{reg}}$ denote the subset of Y_T consisting of pairs (z, t) where $t \in T_{\mathrm{reg}}$. Y_{reg} and $Y_{T,\mathrm{reg}}$ are dense, open subsets of Y and Y_T respectively. Also, the covering map $p_{\mathrm{reg}}\colon G/T \times T_{\mathrm{reg}} \to G_{\mathrm{reg}}$ pulls back along the projection $Y_{\mathrm{reg}} \to G_{\mathrm{reg}}$ to define a covering map $G/T \times Y_{T,\mathrm{reg}} \to Y_{\mathrm{reg}}$. This is just the restriction of the map p_Y defined above. Now the same argument used to prove Corollary 6.2 applies to prove the following Lemma.

Lemma 6.3. *The map $p_Y\colon G/T \times Y_T \to Y$ induces by pullback an injective map on differential forms*

$$(p_Y)^*\colon \Omega(Y) \to \Omega(G/T \times Y_T).$$

One can prove in exactly the same way that the map $G/T \times Y_T^{[2]} \to Y^{[2]}$ induces an injective pullback map on differential forms. Here the fibre products $Y_T^{[2]}$ and $Y^{[2]}$ are formed with respect to the projections $Y_T \to T$ and $Y \to G$ respectively.

For later use, we will calculate the pullback $p^*(g^{-1}dg)$ of the operator valued 1-form $g^{-1}dg$ to $G/T \times Y_T$. We get

$$p^*(g^{-1}dg) = \sum_{\lambda_i} \lambda_i^{-1} d\lambda_i P_i + \sum_{\lambda_i, \lambda_j} \lambda_i^{-1} \lambda_j P_i dP_j. \tag{6.1}$$

We finish this section with a remark about the eigenvalues of a unitary matrix. Clearly any unitary matrix g can be written in the form $\sum_i \lambda_i P_i$ and it is even true that the eigenvalues λ_i depend continuously on g, since they are the solutions of the equation $\det(g - \lambda 1) = 0$ and hence vary continuously with g. Consider the open subset U_z of G consisting of all unitary matrices g such that z is not an eigenvalue of g. We could then define a partial order on the unit circle $U(1) \setminus \{z\}$ in the same manner as in Section 3 above. We can then order the eigenvalues $\lambda_i(g)$ of the unitary matrices g belonging to U_z and we may wonder whether the λ_i depend *smoothly* on g. However this is not the case as the following example shows. Take $G = SU(2)$ and consider the open set U_i consisting of those $g \in SU(2)$ for which i is not an eigenvalue. Consider the intersection

$$U_i \cap T = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \mid \alpha \neq i \right\}$$

of U_i with the standard diagonal torus T. Let g_α denote the diagonal matrix above. Then if we define $\lambda_1 \colon U_1 \to U(1)$ to be the first eigenvalue of g_α relative to the ordering of $U(1) \setminus \{i\}$ defined by rotating in a clockwise direction from i then the value of λ_1 on $U_i \cap T$ is

$$\lambda_1(g_\alpha) = \begin{cases} \alpha & \text{if } x > 0, y > 0 \\ \bar{\alpha} & \text{if } x < 0, y > 0 \\ \alpha & \text{if } x < 0, y < 0 \\ \bar{\alpha} & \text{if } x > 0, y < 0 \end{cases}$$

with $\alpha = x + iy$. This is continuous but not differentiable at $\alpha = \pm 1$. In order to evaluate the various contour integrals we consider it is necessary for us to be able to write g in the form $\sum_i \lambda_i P_i$. Passing to the space $G/T \times T$ avoids this potential problem with the lack of smooth dependence on g of the eigenvalues $\lambda_i(g)$.

7. The basic bundle gerbe as an equivariant bundle gerbe

Suppose that a compact Lie group K acts smoothly on a manifold M, and that (L, Y) is a bundle gerbe over M.

Definition 7.1. We will say that (L, Y) is a K-*equivariant bundle gerbe*[1] if the following conditions are satisfied:

(1) there is an action of K on Y for which the surjective submersion $\pi \colon Y \to M$ is a K-equivariant map,
(2) the line bundle $L \to Y^{[2]}$ is a K-equivariant line bundle for the induced K-action on $Y^{[2]}$.

[1] It is possible to consider a weaker notion of K-equivariant bundle gerbe — see for instance [18]. The simpler notion that we describe here will be sufficient for our purposes.

(3) the section s of the line bundle $\delta(L)$ on $Y^{[3]}$ is K-equivariant.

For more information on equivariant gerbes the reader is referred to [2, 18, 27].

In the introduction we discussed the various realisations of the basic bundle gerbe that have appeared in the literature so far. In the constructions [3, 6, 18] it is proven that the basic bundle gerbe is an equivariant bundle gerbe for the action of G on itself by conjugation. This is also true for our realisation of the basic bundle gerbe. Let G denote one of the unitary groups described in Section 3.

Proposition 7.2. *Let G act on itself by conjugation. Then the basic bundle gerbe on G constructed in Section 3 is an equivariant bundle gerbe in the sense of Definition 7.1 above for this G action.*

We first note that the conjugation action on G lifts to an action on Y: if $(z, g) \in Y$ and $k \in G$ then $(z, kgk^{-1}) \in Y$ since conjugation does not change the eigenvalues of a unitary operator. This G-action naturally induces one on $Y^{[2]}$ and we need to check that the line bundle L is equivariant for this G-action. Recall that $L \to Y_+^{[2]}$ is defined to be the top exterior power

$$L = \det(E)$$

where E is the vector bundle on $Y_+^{[2]}$ defined by the projection valued map

$$P\colon Y^{[2]} \to B(H)$$

G acts naturally on H and hence on $B(H)$; if X is a bounded operator on H, then $k(X)$ is the bounded operator given by $k(X)(v) = kX(k^{-1}v)$ for v a vector in H. Therefore, to prove that $L \to Y_+^{[2]}$ is equivariant, it is enough to prove that E is an equivariant vector bundle, and to do that it is enough to prove that P is an equivariant map.

The value of P at a point (z_1, z_2, g) is the orthogonal projection onto the subspace

$$E_{(z_1,z_2,g)} = \bigoplus_{z_1>\lambda>z_2} E_{(g,\lambda)}$$

If v is an eigenvector of g with eigenvalue λ then clearly $k(v)$ is an eigenvector of kgk^{-1} with eigenvalue λ. Since G acts as a group of unitary operators on H, it follows that $P(z_1, z_2, kgk^{-1}) = kP(z_1, z_2, g)k^{-1}$, i.e P is equivariant. It is now clear how to extend the G-action on the line bundle L over $Y_+^{[2]}$ to a G-action on L over the entire space $Y^{[2]}$. It is also easy to see that the section s of $\delta(L)$ defining the bundle gerbe product on L is equivariant for these G-actions. This completes the proof.

Notice in particular that Proposition 7.2 implies that if T is the diagonal torus inside $G = U(n)$ then (L_T, Y_T) is a T-equivariant bundle gerbe for the trivial action of T on itself by conjugation. It is interesting to study the pullback of (L, Y) along the map $p\colon G/T \times T$; in some sense the bundle gerbe (L, Y) 'abelianises'. To understand what we mean by this observe that since T is a subgroup of G there is a canonical way to extend the T-equivariant bundle gerbe (L_T, Y_T) on T to a G-equivariant gerbe on $G/T \times T$. We make Y_T into a G-space by forming $G \times^T Y_T = G/T \times Y_T$, and we make L_T into a G-equivariant line bundle over $G/T \times Y_T^{[2]}$ by forming $G \times^T L_T$. The pair $(G \times^T L_T, G/T \times Y_T)$ is a G-equivariant bundle gerbe on $G/T \times T$. We have the following proposition.

Proposition 7.3. *There is a bundle gerbe morphism*

$$(G \times^T L_T, G/T \times Y_T, G/T \times T) \to (L, Y, G)$$

covering the map $p\colon G/T \times T \to G$. *In particular the bundle gerbe* p^*L *on* $G/T \times T$ *has the same Dixmier-Douady class as* $(G \times^T L_T, G/T \times Y_T)$.

Here by a morphism of bundle gerbes we understand a morphism in the sense of [19]. As we have already noted above, there is a canonical G-equivariant map $p_Y\colon G/T \times Y_T \to Y$ covering the Weyl map $p\colon G/T \times T \to G$, which sends a pair $(gT, (z, t)) \in G/T \times Y_T$ to the pair $p_Y(gT, (z, t)) = (z, gtg^{-1}) \in Y$.

There is also a canonical G-equivariant map $G \times^T L_T \to L$ covering the induced map $p_Y^{[2]}\colon G/T \times Y_T^{[2]} \to Y^{[2]}$. This is defined as follows. Suppose that $(z_1, z_2, t) \in Y_{T,+}^{[2]}$, where $Y_{T,+}^{[2]}$ is defined in the analogous manner to $Y_+^{[2]}$. Let $v \in E_{(z_1,z_2,t)}$ be an eigenvector of t corresponding to some eigenvalue $\lambda \in (z_1, z_2)$. If $g \in G$ then gv is an eigenvector of gtg^{-1} corresponding to the eigenvalue λ. In other words $gv \in E_{(z_1,z_2,gtg^{-1})}$. We have therefore a linear map

$$E_{(z_1,z_2,t)} \to E_{(z_1,z_2,gtg^{-1})}.$$

On taking top exterior powers this gives a linear map

$$(L_T)_{(z_1,z_2,t)} \to L_{(z_1,z_2,gtg^{-1})}.$$

These linear maps are the restrictions to the fibres of a morphism of line bundles $\hat{p}\colon G \times^T L_T \to L$, covering $p_Y^{[2]}\colon G \times^T Y_{T,+}^{[2]} \to Y_+^{[2]}$. It is clear that moreover this morphism is G-equivariant. By duality we get a corresponding G-equivariant morphism of line bundles covering $p_Y^{[2]}\colon G \times^T Y_{T,-}^{[2]} \to Y_-^{[2]}$ Trivially a similar statement is true over $Y_{T,0}^{[2]}$. Hence we have a G-equivariant morphism of G-equivariant line bundles $\hat{p}\colon G \times^T L_T \to L$ which covers $p_Y^{[2]}$. It is not difficult to show that $\hat{p}$ respects the bundle gerbe products on $G \times^T L_T$ and L. The triple

$$(\hat{p}, p_Y, p)\colon (G \times^T L_T, G/T \times Y_T, G/T \times T) \to (L, Y, G)$$

is a morphism of bundle gerbes in the sense of [19].

Appendix A. Proof of Proposition 4.1

As in Section 3 above, if $(z_1, z_2, g) \in Y_+^{[2]}$ we let $P = P_{(z_1,z_2,g)}$ denote the orthogonal projection onto the eigenspaces $E_{(g,\lambda)}$ for $\lambda \in (z_1, z_2)$. We have the contour integral expression (3.4) for the restriction of P to the open set $U_{(w_1,w_2,h)} = U_{w_1} \times U_{w_2} \times U_h$ associated to a point $(w_1, w_2, h) \in Y_+^{[2]}$:

$$P = \frac{1}{2\pi i} \oint_C (\xi - g)^{-1} d\xi$$

Here C is the contour in $U_{(w_1,w_2,h)}$ described earlier. This formula clearly shows that P is differentiable, since the resolvent $(\xi - g)^{-1}$ depends smoothly on g. It is not hard to show that the derivative dP of P at a point (z_1, z_2, g) in the open set $U_{(w_1,w_2,h)}$ is given by the expression

$$dP = \frac{1}{2\pi i} \oint_C (\xi - g)^{-1} dg (\xi - g)^{-1} d\xi$$

where as above, dg denotes the derivative at (z_1, z_2, g) of the projection of $Y^{[2]}$ onto G. Note that the integrand is an analytic function of ξ and hence we can deform

the contour C without changing the value of the integral. In particular we may write

$$dP = \frac{1}{2\pi i}\oint_{C_{(z_1,z_2,g)}} (\xi - g)^{-1} dg (\xi - g)^{-1} d\xi$$

where for any $(z_1, z_2, g) \in Y_+^{[2]}$, the contour $C_{(z_1,z_2,g)}$ is chosen so that it surrounds the part of the spectrum of g lying between z_1 and z_2. We now compute the expression for the curvature

$$F_{\det(\nabla)} = \operatorname{tr}(PdPdP)$$

in terms of these contour integral formulas. First of all choose contours $C_{(z_1,z_2,g)}$, $C'_{(z_1,z_2,g)}$ and $C''_{(z_1,z_2,g)}$ surrounding the part of the spectrum of g lying in the set (z_1, z_2). We can suppose that the contours are nested in the sense that $C_{(z_1,z_2,g)}$ is contained inside $C'_{(z_1,z_2,g)}$ which is contained inside $C''_{(z_1,z_2,g)}$. Then $F_{\det(\nabla)}$ can be written as an iterated contour integral

$$\left(\frac{1}{2\pi i}\right)^3 \oint_{C_{(z_1,z_2,g)}} \oint_{C'_{(z_1,z_2,g)}} \oint_{C''_{(z_1,z_2,g)}} \operatorname{tr}\left(R(\xi,\eta,\zeta,g)\right) d\xi d\eta d\zeta$$

where $R(\xi, \eta, \zeta, g)$ is the product of resolvents

$$R(\xi,\eta,\zeta,g) = (\xi - g)^{-1}(\eta - g)^{-1} dg (\eta - g)^{-1}(\zeta - g)^{-1} dg (\zeta - g)^{-1}$$

Using the resolvent formula we can rewrite the products appearing in the integrand as differences, for example

$$(\xi - g)^{-1}(\eta - g)^{-1} = (\eta - \xi)\left[(\xi - g)^{-1} - (\eta - g)^{-1}\right].$$

Using this fact and the cyclic property of the trace we can rewrite the integrand as $(\zeta - \eta)^{-1}(\zeta - \xi)^{-1}(\eta - \xi)^{-1}$ times the two-form

$$\operatorname{tr}\left(\left[(\xi - g)^{-1} - (\eta - g)^{-1}\right] dg \left[(\eta - g)^{-1} - (\zeta - g)^{-1}\right] dg\right)$$

Expanding out the product inside the trace leaves us with four contour integrals to compute. Of these, the only non-vanishing contour integral gives

$$\left(\frac{1}{2\pi i}\right)^3 \oint_{C_{(z_1,z_2,g)}} \oint_{C'_{(z_1,z_2,g)}} \oint_{C''_{(z_1,z_2,g)}} \frac{\operatorname{tr}\left((\xi - g)^{-1} dg (\zeta - g)^{-1} dg\right)}{(\zeta - \eta)(\zeta - \xi)(\eta - \xi)} d\xi d\eta d\zeta$$

which we can evaluate to

$$\text{(A.1)} \qquad \left(\frac{1}{2\pi i}\right)^2 \oint_{C_{(z_1,z_2,g)}} \oint_{C''_{(z_1,z_2,g)}} (\zeta - \xi)^{-2} \operatorname{tr}\left((\xi - g)^{-1} dg (\zeta - g)^{-1} dg\right) d\xi d\zeta.$$

The function $(\zeta - \xi)^{-2}$ is a holomorphic function of ξ, since $\zeta \in C''_{(z_1,z_2,g)}$ which lies outside $C_{(z_1,z_2,g)}$. Therefore we can simplify this expression further using the holomorphic functional calculus. Since $(\zeta - \xi)^2(\zeta - \xi)^{-2} = 1$ we have (using the property $f(T)f'(T) = (ff')(T)$ of the functional calculus)

$$\frac{1}{2\pi i}\oint_{C_{(z_1,z_2,g)}} (\zeta - \xi)^{-2}(\xi - g)^{-1} d\xi = (\zeta - g)^{-2} P_{z_1 z_2}.$$

Therefore, after renaming variables, we can rewrite (A.1) above as the single contour integral

$$\frac{1}{2\pi i}\oint_{C_{(z_1,z_2,g)}} \operatorname{tr}\left((\xi - g)^{-2} P_{z_1 z_2} dg (\xi - g)^{-1} dg\right) d\xi.$$

We want to show that we have the equality of Proposition 4.1:

$$\begin{aligned}\text{(A.2)}\quad \frac{1}{2\pi i}\oint_{C_{(z_1,z_2,g)}} \operatorname{tr}\left((\xi-g)^{-1}dg(\xi-g)^{-2}P_{z_1z_2}dg\right)d\xi \\ = \frac{1}{4\pi i}\oint_{C_{(z_1,z_2,g)}} \operatorname{tr}\left((\xi-g)^{-1}dg(\xi-g)^{-2}dg\right)d\xi\end{aligned}$$

Up to now everything that we have said works equally well for G finite dimensional or for G one of the infinite dimensional groups $U_p(H)$. Let us now suppose that the group G is the finite dimensional unitary group $U(H)$, for H a finite dimensional complex Hilbert space. Thus G is isomorphic to $U(n)$ where n is the dimension of H. This is the first instance where we will take advantage of the special properties of the map $p\colon G/T\times T\to G$. From the remark following Lemma 6.3 we have that the pullback map $\Omega(Y^{[2]})\to\Omega(G/T\times Y_T)$ induced by $G/T\times Y_T^{[2]}\to Y^{[2]}$ is injective. If we think of a point of $G/T\times Y_T^{[2]}$ as a family $(P_i,(z_1,z_2,\lambda_i))$ then the map $G/T\times Y_T^{[2]}\to Y^{[2]}$ can be written as

$$(P_i,(z_1,z_2,\lambda_i))\mapsto (z_1,z_2,\sum\lambda_iP_i)$$

Using this description of the map $G/T\times Y_T^{[2]}\to Y^{[2]}$ in terms of the projections P_i and eigenvalues λ_i we see that we can write the pullback by p of the left hand side of (A.2) as

$$\frac{1}{2\pi i}\oint_{C_{(z_1,z_2,g)}}\sum_{i,j}(\xi-\lambda_i)^{-1}(\xi-\lambda_j)^{-2}\operatorname{tr}\left(P_ip^*(dg)P_jP_{z_1z_2}p^*(dg)\right)d\xi$$

To simplify this expression we will make use of the following easily proven facts:

$$\begin{aligned}&\operatorname{Res}_{\xi=\lambda_j}(\xi-\lambda_i)^{-1}(\xi-\lambda_j)^{-2}=-(\lambda_i-\lambda_j)^{-2}\\&\operatorname{Res}_{\xi=\lambda_i}(\xi-\lambda_i)^{-1}(\xi-\lambda_j)^{-2}=(\lambda_i-\lambda_j)^{-2}\\&P_jP_{z_1z_2}=\begin{cases}P_j & \text{if } j\in(z_1,z_2)\\ 0 & \text{if } j\notin(z_1,z_2)\end{cases}\end{aligned}$$

By splitting the sum over the eigenvalues λ_i into the sum over the eigenvalues $\lambda_i\in(z_1,z_2)$ and the eigenvalues $\lambda_i\notin(z_1,z_2)$ and using the above facts, we derive the following expression for the pullback by p of the left hand side of (A.2):

$$-\sum_{\substack{\lambda_i\notin(z_1,z_2)\\ \lambda_j\in(z_1,z_2)}}(\lambda_i-\lambda_j)^{-2}\operatorname{tr}(P_ip^*(dg)P_jp^*(dg))$$

On the other hand, it is easy to see that the pullback by p of the right hand side of (A.2) can be written as

$$\frac{1}{4\pi i}\oint_{C_{(z_1,z_2,g)}}\sum_{i,j}(\xi-\lambda_i)^{-1}(\xi-\lambda_j)^{-2}\operatorname{tr}(P_ip^*(dg)P_jp^*(dg))d\xi$$

Again, we can split the sum over eigenvalues λ_i and λ_j into sums over eigenvalues belonging to the sets (z_1, z_2) or their complements to get

$$\frac{1}{4\pi i}\oint_{C_{(z_1,z_2,g)}} \sum_{\substack{\lambda_i\in(z_1,z_2)\\ \lambda_j\in(z_1,z_2)}} (\xi-\lambda_i)^{-1}(\xi-\lambda_j)^{-2}\,\mathrm{tr}(P_i p^*(dg)P_j p^*(dg))d\xi$$
$$+\frac{1}{4\pi i}\oint_{C_{(z_1,z_2,g)}} \sum_{\substack{\lambda_i\notin(z_1,z_2)\\ \lambda_j\in(z_1,z_2)}} (\xi-\lambda_i)^{-1}(\xi-\lambda_j)^{-2}\,\mathrm{tr}(P_i p^*(dg)P_j p^*(dg))d\xi$$
$$+\frac{1}{4\pi i}\oint_{C_{(z_1,z_2,g)}} \sum_{\substack{\lambda_i\in(z_1,z_2)\\ \lambda_j\notin(z_1,z_2)}} (\xi-\lambda_i)^{-1}(\xi-\lambda_j)^{-2}\,\mathrm{tr}(P_i p^*(dg)P_j p^*(dg))d\xi$$

By the residue theorem these contour integrals become

$$\begin{aligned}
&-\frac{1}{2}\sum_{\substack{\lambda_i\notin(z_1,z_2)\\ \lambda_j\in(z_1,z_2)}} (\lambda_i-\lambda_j)^{-2}\,\mathrm{tr}\,(P_i p^*(dg)P_j p^*(dg))\\
&+\frac{1}{2}\sum_{\substack{\lambda_i\in(z_1,z_2)\\ \lambda_j\notin(z_1,z_2)}} (\lambda_i-\lambda_j)^{-2}\,\mathrm{tr}\,(P_i p^*(dg)P_j p^*(dg))\\
&=-\sum_{\substack{\lambda_i\notin(z_1,z_2)\\ \lambda_j\in(z_1,z_2)}} (\lambda_i-\lambda_j)^{-2}\,\mathrm{tr}\,(P_i p^*(dg)P_j p^*(dg))
\end{aligned}$$

and so we see the two expressions are equal. It follows by the injectivity of the map $p^*\colon \Omega^2(Y^{[2]}) \to \Omega^2(G/T\times Y_T^{[2]})$ that the two forms on $Y^{[2]}$ are equal.

Now let us suppose that G is one of the infinite dimensional Banach Lie groups $U_p(H)$ for $1 \le p \le 2$. We have a pair of two-forms defined on $Y_+^{[2]}$ and we want to prove that we have the equality of (A.2). Since neither of these two-forms have components in the $U(1)$ directions, it is sufficient to prove that the two-forms are equal for fixed z_1 and z_2. Thus we can regard them as forms on the open subset $U_{z_1z_2}$ of G consisting of those $g\in G$ for which neither z_1 nor z_2 is an eigenvalue. To prove that they are equal, we will use a slight modification of the argument used to prove Theorem 5.3 and conclude that if ω is a form defined on an open subset U of G such that ω vanishes on restriction to $U\cap U(H_n)$ for all n, then ω is zero. The equality of (A.2) follows easily from this conclusion.

Recall that Quillen proves Theorem 5.3 using the ‘tame approximation’ theorem of Palais [22], which states that for any $A\in\mathcal{L}_p$, $P_nAP_n\to A$ in $\mathcal{L}_p$. A p-form ω on $GL_p(H)$ can be regarded as a smooth map $\omega(A_1,\dots,A_p)(g)$ which is alternating and multilinear in the $A_i\in\mathcal{L}_p$. If ω vanishes on restriction to $GL(H_n)$ for every n then it follows from the tame approximation theorem that ω vanishes on $GL_p(H)$. More generally if U is an open subset of $GL_p(H)$ and ω is a p-form on U which vanishes on restriction to any $GL(H_n)\cap U$ for every n then $\omega=0$. To prove the theorem for the unitary groups $U_p(H)$ Quillen considers the phase retraction map

$$\begin{aligned}
&GL_p(H)\to U_p(H)\\
&g\mapsto g(gg^*)^{-1/2}.
\end{aligned}$$

It is easy to see that this maps the subgroups $GL(H_n)$ of $GL_p(H)$ to the corresponding subgroups $U(H_n)$ of $U_p(H)$. The phase retraction has the property that it maps forms on $U_p(H)$ injectively to forms on $GL_p(H)$. The result for $U_p(H)$ now follows from the corresponding result for $GL_p(H)$. Again let us note that if ω is a p-form defined on an open subset $U \subset U_p(H)$ such that ω vanishes on restriction to any $U(H_n) \cap U$, then ω vanishes on U. To see this, note that if V is the inverse image of U under the phase retraction map, then the inverse image of $U(H_n) \cap U$ is mapped to $GL(H_n) \cap V$. If ω' denotes the pullback of ω to V, then ω' vanishes on restriction to any $GL(H_n) \cap V$. Therefore ω', and hence ω, is zero.

Appendix B. Proof of Theorem 5.1 part (b).

The map $p_Y \colon G/T \times Y_T \to Y$ from Lemma 6.3 fits into a commutative diagram

$$\begin{tikzcd} G/T \times Y_T \arrow[r] \arrow[d, "p_Y"'] & G/T \times T \arrow[d, "p"] \\ Y \arrow[r, "\pi"] & G \end{tikzcd}$$

where we recall that we write π for the projection map $Y \to G$. We want to show that $df = \pi^*(2\pi i\nu)$ where ν is the basic three-form (1.1). It is sufficient to show this equality on the dense open subset $Y_{\text{reg}} \subset Y$. $(p_Y)^*$ is then injective on $\Omega(Y_{\text{reg}})$ since $p_Y \colon G/T \times Y_{T,\text{reg}} \to Y_{\text{reg}}$ is a covering map. Note that by commutativity of the diagram, $(p_Y)^*\pi^*(2\pi i\nu)$ is equal to the pullback of $p^*(2\pi i\nu)$ along the map $G/T \times Y_T \to G/T \times T$. We begin with the left hand side of the equation $(p_Y)^*df = (p_Y)^*\pi^*(2\pi i\nu)$. We first calculate the pullback $(p_Y)^*f$ of the restriction of f to Y_{reg}. Note that a point in $G/T \times Y_{T,\text{reg}}$ is of the form $((P,\lambda), z)$ where $(P,\lambda) = (P_i, \lambda_i)$ is a family of projections P_i and eigenvalues λ_i and where z is not equal to any of the λ_i. Since $(P,\lambda) \in G/T \times T_{\text{reg}}$, all of the λ_i are *distinct*. Thus if $i \neq j$ then $P_iP_j = P_jP_i = 0$. The pullback $(p_Y)^*f$ is given by the contour integral

$$\frac{1}{8\pi^2}\oint_{C_{(z,g)}} \sum \log_z \xi(\xi-\lambda_i)^{-1}(\xi-\lambda_k)^{-2}\operatorname{tr}(P_i(d\lambda_jP_j + \\ + \lambda_j dP_j)P_k(d\lambda_lP_l + \lambda_l dP_l))d\xi$$

We simplify the term within the trace summed over j and l. Expanding it out and reindexing gives us

$$\begin{aligned} &\sum_{j,l}\operatorname{tr}(d\lambda_j\lambda_l\delta_{ij}\delta_{jk}P_jdP_l - \lambda_jd\lambda_i\delta_{kl}\delta_{li}P_idP_j + \lambda_j\lambda_lP_idP_jP_kdP_l) \\ &\qquad = \sum_l \operatorname{tr}(d\lambda_i\lambda_l\delta_{ik}P_idP_l) - \sum_j\operatorname{tr}(\lambda_jd\lambda_i\delta_{ik}P_idP_j) + \sum_{j,l}\operatorname{tr}(\lambda_j\lambda_lP_idP_jP_kdP_l) \\ &\qquad = \sum_j \operatorname{tr}(\lambda_j\lambda_lP_idP_jP_kdP_l) \end{aligned}$$

Inserting this expression back into the contour integral gives us

$$\frac{1}{8\pi^2}\oint_{C_{(z,g)}}\sum\log_z\xi(\xi-\lambda_i)^{-1}(\xi-\lambda_k)^{-2}\lambda_j\lambda_l\operatorname{tr}(P_idP_jP_kdP_l)d\xi. \tag{B.1}$$

To evaluate this contour integral we use Cauchy's residue theorem. For each pair of indices i and k we need to calculate the residues of the function $\log_z \xi(\xi-\lambda_i)^{-1}(\xi-\lambda_k)^{-2}$ at the poles $\xi=\lambda_i$ and $\xi=\lambda_k$. When $i\neq k$ we have

$$\mathrm{Res}_{\xi=\lambda_i}(\xi-\lambda_i)^{-1}(\xi-\lambda_k)^{-2}\log_z\xi=\log_z\lambda_i(\lambda_i-\lambda_k)^{-2}$$

and

$$\mathrm{Res}_{\xi=\lambda_k}(\xi-\lambda_i)^{-1}(\xi-\lambda_k)^{-2}\log_z\xi=(\lambda_k-\lambda_i)^{-1}\lambda_k^{-1}-\log_z\lambda_k(\lambda_k-\lambda_i)^{-2}.$$

We also need to consider the residue at the pole $\xi=\lambda_i=\lambda_k$ in the case where $i=k$. In this case however, the two-form $\mathrm{tr}(P_i dP_j P_i dP_k)$ vanishes. This is clear when $j=i$ as $P_i dP_i P_i=0$. When $j\neq i$ we can use the identity $dP_i P_j+P_i dP_j=0$ to write $P_i dP_j P_i=-dP_i P_j P_i=0$. Therefore, only the residues at the poles $\xi=\lambda_i$ and $\xi=\lambda_k$ for i and k distinct give contributions. Therefore (B.1) becomes

$$\frac{i}{4\pi}\sum_{i\neq k}\big(\log_z\lambda_i(\lambda_i-\lambda_k)^{-2}-\log_z\lambda_k(\lambda_i-\lambda_k)^{-2} \\ +(\lambda_k-\lambda_i)^{-1}\lambda_i^{-1}\big)\,\lambda_j\lambda_l\,\mathrm{tr}(P_i dP_j P_k dP_l) \tag{B.2}$$

To simplify this expression we consider, for fixed λ_i and λ_k, the sum

$$\sum_{j,l}\lambda_j\lambda_l\,\mathrm{tr}(P_i dP_j P_k dP_l) \tag{B.3}$$

To evaluate this sum we shall make use of the following identity:

$$\sum_j\lambda_j P_i dP_j P_k=\lambda_i P_i dP_i P_k+\lambda_k P_i dP_k P_k.$$

To see this note we write the sum as

$$\sum_j\lambda_j P_i dP_j P_k=\lambda_i P_i dP_i P_k+\lambda_k P_i dP_k P_k+\sum_{i\neq j,j\neq k}\lambda_j P_i dP_j P_k$$

If the indices i, j and k are such that $i\neq j$ and $j\neq k$, then we can write

$$P_i dP_j P_k=-dP_i P_j P_k=0,$$

and hence the identity. Applying this identity we find that (B.3) becomes a sum of two terms:

$$\sum_l\lambda_i\lambda_l\,\mathrm{tr}(P_i dP_i P_k dP_l)+\sum_l\lambda_k\lambda_l\,\mathrm{tr}(P_i dP_k P_k dP_l)$$

We can apply the identity again to each of the terms in this new sum, after first writing $\mathrm{tr}(P_i dP_i P_k dP_l)=\mathrm{tr}(P_i dP_i P_k dP_l P_i)$ and $\mathrm{tr}(P_i dP_k P_k dP_l)=\mathrm{tr}(P_i dP_k P_k dP_l P_i)$. We obtain

$$\lambda_i\lambda_k\,\mathrm{tr}(P_i dP_i P_k dP_k)+\lambda_i^2\,\mathrm{tr}(P_i dP_i P_k dP_i) \\ +\lambda_k^2\,\mathrm{tr}(P_i dP_k P_k dP_k)+\lambda_i\lambda_k\,\mathrm{tr}(P_i dP_k P_k dP_i).$$

Writing $dP_k=dP_k P_k+P_k dP_k$, $dP_i=dP_i P_i+P_i dP_i$ and using the identities $P_i P_k=\delta_{ik}P_i$ and $dP_k P_i=\delta_{ik}dP_k-P_k dP_i$, we see that we can write this as

$$(1-\delta_{ik})\lambda_i\lambda_k\,\mathrm{tr}(P_i dP_i dP_k)-(1-\delta_{ik})\lambda_i^2\,\mathrm{tr}(P_k dP_i dP_i) \\ +(1-\delta_{ik})\lambda_k^2\,\mathrm{tr}(P_i dP_k dP_k)-(1-\delta_{ik})\lambda_i\lambda_k\,\mathrm{tr}(P_i dP_k dP_k).$$

We can further simplify this, using the above properties of the projections P_i and P_k, to

$$(1-\delta_{ik})(\lambda_i-\lambda_k)^2\operatorname{tr}(P_i dP_k dP_k) = (\lambda_i-\lambda_k)^2\operatorname{tr}(P_i dP_k dP_k).$$

Substituting this into (B.2) gives the following expression for the pullback $(p_Y)^*f$:

$$\frac{i}{4\pi}\sum_{i\neq k}\left(\log_z\lambda_i - \log_z\lambda_k + (\lambda_k-\lambda_i)\lambda_k^{-1}\right)\operatorname{tr}(P_i dP_k dP_k). \tag{B.4}$$

On taking the exterior derivative we end up with the following expression for $d(p_Y)^*f$:

$$\begin{aligned}\frac{i}{4\pi}\sum_{i\neq k}&\left(\lambda_i^{-1}d\lambda_i - \lambda_k^{-1}d\lambda_k - d\lambda_i\lambda_k^{-1} + \lambda_i\lambda_k^{-1}d\lambda_k\lambda_k^{-1}\right)\operatorname{tr}(P_i dP_k dP_k)\\ &+\frac{i}{4\pi}\sum_{i\neq k}\left(\log_z\lambda_i - \log_z\lambda_k + 1 - \lambda_i\lambda_k^{-1}\right)\operatorname{tr}(dP_i dP_k dP_k).\end{aligned} \tag{B.5}$$

We can simplify the second term in this expression further: since $\operatorname{tr}(dP_i dP_i dP_i) = 0$ we see that we can write $\sum_{i\neq k}\operatorname{tr}(dP_i dP_k dP_k) = \sum_{i,k}\operatorname{tr}(dP_i dP_k dP_k)$ and this last sum is zero since $\sum dP_i = 0$. We can also reindex and write

$$\begin{aligned}\sum_{i\neq k}\log_z\lambda_i&\operatorname{tr}(dP_i dP_k dP_k) - \sum_{i\neq k}\log_z\lambda_k\operatorname{tr}(dP_i dP_k dP_k)\\ &= \sum_{i\neq k}\log_z\lambda_i\operatorname{tr}(dP_i dP_k dP_k) - \sum_{i\neq k}\log_z\lambda_i\operatorname{tr}(dP_k dP_i dP_i)\end{aligned}$$

If $i\neq k$ then it is easy to see that $\operatorname{tr}(dP_i dP_k dP_k) = -\operatorname{tr}(dP_k dP_i dP_i)$. Therefore we can write the above expression as

$$-2\sum_{i\neq k}\log_z\lambda_i\operatorname{tr}(dP_k dP_i dP_i).$$

Again, since $\operatorname{tr}(dP_i dP_i dP_i) = 0$ we can write this as

$$-2\sum_{i,k}\log_z\lambda_i\operatorname{tr}(dP_k dP_i dP_i)$$

which equals zero since $\sum dP_k = 0$. Therefore we end up with the following expression for $(p_Y)^*df$:

$$\begin{aligned}\frac{i}{4\pi}\sum_{i\neq k}&\left(\lambda_i^{-1}d\lambda_i - \lambda_k^{-1}d\lambda_k - d\lambda_i\lambda_k^{-1} + \lambda_i\lambda_k^{-1}d\lambda_k\lambda_k^{-1}\right)\operatorname{tr}(P_i dP_k dP_k)\\ &-\frac{i}{4\pi}\sum_{i\neq k}\lambda_i\lambda_k^{-1}\operatorname{tr}(dP_i dP_k dP_k).\end{aligned} \tag{B.6}$$

We would now like to compare this expression with the pullback three-form $p^*(2\pi i\nu)$ where we remind the reader again that ν is the three-form (1.1). To compute this recall from equation (6.1) that we had

$$p^*(g^{-1}dg) = \lambda_i^{-1}d\lambda_i P_i + \lambda_i^{-1}\lambda_j P_i dP_j.$$

After a little calculation one finds that the pullback three-form $2\pi i p^*\nu$ is given by the following sum of two terms

(B.7)
$$-\frac{i}{4\pi}\operatorname{tr}\left(\lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_j\lambda_k^{-1}\lambda_l P_i dP_j P_k dP_l\right)-\frac{i}{12\pi}\frac{\lambda_j\lambda_l\lambda_n}{\lambda_i\lambda_k\lambda_m}\operatorname{tr}\left(P_i dP_j P_k dP_l P_m dP_n\right)$$

where in each term there is understood to be a sum over all appropriate indices. We concentrate on the first term in (B.7) to begin with (for clarity we omit the factor of $-i/4\pi$). Making use of the fact that $dP_iP_j + P_idP_j = \delta_{ij}dP_i$ (so that $P_idP_j = dP_i(\delta_{ij} - P_j)$) we see that we can write it as

$$\begin{aligned}
&\sum \lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_j\lambda_k^{-1}\lambda_l\operatorname{tr}(P_idP_jP_kdP_l)\\
=&\sum \lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_j\lambda_k^{-1}\lambda_l\operatorname{tr}(\delta_{ij}dP_i - dP_iP_j)P_kdP_l)\\
=&\sum\left(\left[\lambda_i^{-1}d\lambda_i\lambda_k^{-1}\lambda_l-\lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_l\right]\operatorname{tr}(dP_iP_kdP_l)\right)\\
=&\sum\left(\left[\lambda_i^{-1}d\lambda_i\lambda_k^{-1}\lambda_l-\lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_l\right]\operatorname{tr}\left\{dP_i(\delta_{kl}dP_k-dP_kP_l)\right\}\right)\\
=&\sum\left(\lambda_i^{-1}d\lambda_i\operatorname{tr}(dP_idP_k)-\lambda_i^{-1}d\lambda_i\lambda_k^{-1}\lambda_l\operatorname{tr}(dP_idP_kP_l)\right.\\
&\left.-\lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_k\operatorname{tr}(dP_idP_k)+\lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_l\operatorname{tr}(dP_idP_kP_l)\right)
\end{aligned}$$

We can write the terms $\lambda_i^{-1}d\lambda_i\lambda_k^{-1}\lambda_l\operatorname{tr}(dP_idP_kP_l)$ and $\lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_l\operatorname{tr}(dP_idP_kP_l)$ as

$$d\lambda_i\lambda_k^{-1}\operatorname{tr}(dP_idP_kP_i)+\lambda_i^{-1}d\lambda_i\operatorname{tr}(dP_idP_kP_k)+\sum_{i\neq l,k\neq l}\lambda_i^{-1}d\lambda_i\lambda_k^{-1}\lambda_l\operatorname{tr}(dP_idP_kP_l)$$

and

$$\begin{aligned}
&\lambda_i^{-1}d\lambda_i\operatorname{tr}(dP_idP_kP_i)+\lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_k\operatorname{tr}(dP_idP_kP_k)\\
&\qquad\qquad+\sum_{i\neq l,k\neq l}\lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_l\operatorname{tr}(dP_idP_kP_l).
\end{aligned}$$

respectively. Note that if $i \neq l$ and $k \neq l$ then $\operatorname{tr}(dP_idP_kP_l) = -\delta_{ik}\operatorname{tr}(P_idP_ldP_l)$. This is because

$$\operatorname{tr}(dP_idP_kP_l) = \operatorname{tr}(P_ldP_idP_kP_l) = \operatorname{tr}(dP_lP_iP_kdP_l)$$

It follows that

$$\sum_{i\neq l,k\neq l}\left(\lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_l-\lambda_i^{-1}d\lambda_i\lambda_k^{-1}\lambda_l\right)\operatorname{tr}(dP_idP_kP_l)=0.$$

Therefore we end up with the following expression:

(B.8)
$$\begin{aligned}
\sum&\left(\lambda_i^{-1}d\lambda_i\operatorname{tr}(dP_idP_k)-\lambda_i^{-1}d\lambda_i\operatorname{tr}(dP_idP_kP_k)-d\lambda_i\lambda_k^{-1}\operatorname{tr}(dP_idP_kP_i)\right.\\
&-\lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_k\operatorname{tr}(dP_idP_k)+\lambda_i^{-1}d\lambda_i\operatorname{tr}(dP_idP_kP_i)\\
&\left.+\lambda_i^{-1}d\lambda_i\lambda_i^{-1}\lambda_k\operatorname{tr}(dP_idP_kP_k)\right).
\end{aligned}$$

Using $dP_k = P_kdP_k + dP_kP_k$ we can simplify the terms

$$\sum\left(\lambda_i^{-1}d\lambda_i\operatorname{tr}(dP_idP_k)-\lambda_i^{-1}d\lambda_i\operatorname{tr}(dP_idP_kP_k)\right)$$

appearing in (B.8) to

$$-\sum\lambda_i^{-1}d\lambda_i\operatorname{tr}(P_kdP_kdP_i).$$

Similarly we can simplify the terms

$$\sum \left(\lambda_i^{-1} d\lambda_i \lambda_i^{-1} \lambda_k \operatorname{tr}(dP_i dP_k P_k) - \lambda_i^{-1} d\lambda_i \lambda_i^{-1} \lambda_k \operatorname{tr}(dP_i dP_k)\right)$$

appearing in (B.8) to

$$\sum \lambda_i^{-1} d\lambda_i \lambda_i^{-1} \lambda_k \operatorname{tr}(P_k dP_k dP_i).$$

Therefore (B.8) becomes

$$\begin{aligned}\sum \big(-\lambda_i^{-1} d\lambda_i \operatorname{tr}(P_k dP_k dP_i) - d\lambda_i \lambda_k^{-1} \operatorname{tr}(dP_i dP_k P_i) \\ +\lambda_i^{-1} d\lambda_i \lambda_i^{-1} \lambda_k \operatorname{tr}(P_k dP_k dP_i) + \lambda_i^{-1} d\lambda_i \operatorname{tr}(dP_i dP_k P_i)\big)\end{aligned} \tag{B.9}$$

Replacing dP_k with $dP_k P_k + P_k dP_k$ we can write

$$d\lambda_i \lambda_k^{-1} \operatorname{tr}(dP_i dP_k P_i) = d\lambda_i \lambda_k^{-1} \operatorname{tr}(dP_i P_k dP_k P_i)$$

which is then easily seen to be equal to $-d\lambda_i \lambda_k^{-1} \operatorname{tr}(P_i dP_k dP_k)$. Similarly write

$$\lambda_i^{-1} d\lambda_i \lambda_i^{-1} \lambda_k \operatorname{tr}(P_k dP_k dP_i) = \lambda_i^{-1} d\lambda_i \lambda_i^{-1} \lambda_k \operatorname{tr}(P_k dP_k P_i dP_i)$$

which is in turn equal to $-\lambda_i^{-1} d\lambda_i \lambda_i^{-1} \lambda_k \operatorname{tr}(P_k dP_i dP_i)$. Then after reindexing some of the sums in (B.9) and restoring the factor $-i/4\pi$ we get

$$\frac{i}{4\pi} \sum_{i \neq k} \left(\lambda_i^{-1} d\lambda_i - d\lambda_k \lambda_k^{-1} - \lambda_k^{-1} d\lambda_i + \lambda_k^{-1} d\lambda_k \lambda_k^{-1} \lambda_i\right) \operatorname{tr}(P_i dP_k dP_k) \tag{B.10}$$

where it is clear we lose nothing by restricting the sum to $i \neq k$.

We now turn our attention to the second term appearing in (B.7):

$$\frac{\lambda_j \lambda_l \lambda_n}{\lambda_i \lambda_k \lambda_m} \operatorname{tr}(P_i dP_j P_k dP_l P_m dP_n)$$

Using the fact that $dP_i P_j + P_i dP_j = \delta_{ij} dP_i$ we can write this expression as

$$\begin{aligned}&\operatorname{tr}(\sum \frac{\lambda_j \lambda_l \lambda_n}{\lambda_i \lambda_k \lambda_m} P_i dP_j P_k dP_l P_m dP_n) \\ &= \operatorname{tr}(\sum \frac{\lambda_j \lambda_l \lambda_n}{\lambda_i \lambda_k \lambda_m} (\delta_{mn} - P_n) P_i (\delta_{jk} - P_j) dP_k dP_l dP_m) \\ &= \operatorname{tr}(\sum \frac{\lambda_j \lambda_l \lambda_n}{\lambda_i \lambda_k \lambda_m} (P_i \delta_{mn} \delta_{jk} - \delta_{mn} \delta_{ij} P_j - \delta_{ni} \delta_{jk} P_i + \delta_{ij} \delta_{in} P_j) dP_k dP_l dP_m) \\ &= \operatorname{tr}(\sum \left(\frac{\lambda_k}{\lambda_i} - \frac{\lambda_k}{\lambda_j} - \frac{\lambda_k}{\lambda_l} + \frac{\lambda_i \lambda_k}{\lambda_j \lambda_l}\right) P_i dP_j dP_k dP_l)\end{aligned}$$

We can finally write this as

$$\operatorname{tr}(\sum \left(\frac{\lambda_k (\lambda_l - \lambda_i)(\lambda_j - \lambda_i)}{\lambda_i \lambda_j \lambda_l}\right) P_i dP_j dP_k dP_l) \tag{B.11}$$

This is a sum over four indices i, j, k and l. We make a series of observations to show that only certain combinations of these indices will make non-zero contributions to the sum. This will allow us to eventually greatly simplify the sum.

(1) We must have $i \neq j$ and $i \neq l$. This is clear since $(\lambda_l - \lambda_i)(\lambda_j - \lambda_i)$ is a factor.
(2) The indices i, j, k and l cannot all be distinct. If they were, then using the fact that $P_i P_j = 0$ and hence $dP_i P_j = -P_i dP_j$ we see that the expression $\operatorname{tr}(P_i dP_j dP_k dP_l P_i) = 0$. Therefore some of i, j, k and l must be equal.

(3) No three of the indices i, j, k and l can be equal. There are three possibilities to consider here: $i = j = k$, $i = j = l$ and $j = k = l$. From the first observation above we can exclude the possibility that $i = j = k$ or $i = j = l$. If $j = k = l$ then $\operatorname{tr}(P_i dP_j dP_j dP_j) = -\operatorname{tr}(dP_i dP_j dP_j P_j P_i) = 0$ after using the identities $P_i dP_j = -dP_i P_j$ and $P_j dP_j + dP_j P_j = dP_j$.
(4) We conclude that two and only two of the indices i, j, k and l can be equal. There are now six possibilities to consider (a) $i = j$, (b) $i = k$, (c) $i = l$, (d) $k = l$, (e) $j = l$, (f) $j = k$. Of these six possibilities we can eliminate (a) and (b) immediately from the first observation. The case where $i = k$ also gives no contribution since $\operatorname{tr}(P_d P_j dP_i dP_l) = -\operatorname{tr}(dP_i P_j P_i dP_l dP_i) = 0$. The case where $j = l$ also gives no contribution for a similar sort of reason: $\operatorname{tr}(P_i dP_j dP_k dP_j) = -\operatorname{tr}(dP_i dP_j P_k P_j dP_i) = 0$.

We conclude from this series of observations that we can restrict the sum to the two cases $k = l$ or $j = k$ with all indices otherwise distinct. By reindexing the sum we see that we can write (B.11) as

$$\sum_{i,j,k\,\text{distinct}} \left(\frac{\lambda_j}{\lambda_i} - 1 - \frac{\lambda_j}{\lambda_k} + \frac{\lambda_i}{\lambda_k} \right) \operatorname{tr}(P_i dP_j dP_j dP_k) \\ + \sum_{i,j,k\,\text{distinct}} \left(\frac{\lambda_k}{\lambda_i} - \frac{\lambda_k}{\lambda_j} - 1 + \frac{\lambda_i}{\lambda_j} \right) \operatorname{tr}(P_i dP_j dP_k dP_k)$$

Since $\operatorname{tr}(P_i dP_j dP_k dP_k) = -\operatorname{tr}(P_j dP_k dP_k dP_i)$ for $i \neq j$ we can reindex the second sum to get

$$-\sum_{i,j,k\,\text{distinct}} \left(\frac{\lambda_j}{\lambda_k} - \frac{\lambda_j}{\lambda_i} - 1 + \frac{\lambda_k}{\lambda_i} \right) \operatorname{tr}(P_i dP_j dP_j dP_k).$$

Adding these two sums gives

$$\sum_{i,j,k\,\text{distinct}} \left(2\frac{\lambda_j}{\lambda_i} - 2\frac{\lambda_j}{\lambda_k} + \frac{\lambda_i}{\lambda_k} - \frac{\lambda_k}{\lambda_i} \right) \operatorname{tr}(P_i dP_j dP_j dP_k)$$

The three-forms $\operatorname{tr}(P_i dP_j dP_j dP_k)$ for λ_i, λ_j and λ_k all distinct possess a cyclic symmetry

$$\operatorname{tr}(P_i dP_j dP_j dP_k) = \operatorname{tr}(P_k dP_i dP_i dP_j) = \operatorname{tr}(P_j dP_k dP_k dP_i).$$

Using this cyclic symmetry and re-indexing we get

$$3 \sum_{i,j,k\,\text{distinct}} \left(\frac{\lambda_i}{\lambda_k} - \frac{\lambda_k}{\lambda_i} \right) \operatorname{tr}(P_k dP_i dP_i dP_j).$$

Clearly we can include the terms in this sum with $i = k$ with no effect, so that we can rewrite this as

$$3 \sum_{i \neq j, j \neq k} \left(\frac{\lambda_i}{\lambda_k} - \frac{\lambda_k}{\lambda_i} \right) \operatorname{tr}(P_k dP_i dP_i dP_j).$$

Further, since $\operatorname{tr}(P_i dP_j dP_j dP_j) = 0$ for $i \neq j$ we can write the sum as

$$3 \sum_{j \neq k} \left(\frac{\lambda_i}{\lambda_k} - \frac{\lambda_k}{\lambda_i} \right) \operatorname{tr}(P_k dP_i dP_i dP_j).$$

This in turn can be written as

$$3\sum\left(\frac{\lambda_i}{\lambda_k}-\frac{\lambda_k}{\lambda_i}\right)\operatorname{tr}(P_k dP_i dP_i dP_j) - 3\sum_{i\neq k}\left(\frac{\lambda_i}{\lambda_k}-\frac{\lambda_k}{\lambda_i}\right)\operatorname{tr}(P_k dP_i dP_i dP_k)$$

Again, using the fact that $\sum dP_j = 0$ we end up with

$$-3\sum_{i\neq k}\left(\frac{\lambda_i}{\lambda_k}-\frac{\lambda_k}{\lambda_i}\right)\operatorname{tr}(P_k dP_i dP_i dP_k).$$

Reindexing we can write this as

$$\begin{aligned}-3\sum_{i\neq k}\left(\frac{\lambda_i}{\lambda_k}-\frac{\lambda_k}{\lambda_i}\right)\operatorname{tr}(P_k dP_i dP_i dP_j) + 3\sum_{i\neq k}\frac{\lambda_i}{\lambda_k}\operatorname{tr}(P_i dP_k dP_k dP_i)\\ = -3\sum_{i\neq k}\frac{\lambda_i}{\lambda_k}\left(\operatorname{tr}(P_k dP_i dP_i dP_k) - \operatorname{tr}(P_i dP_k dP_k dP_i)\right)\end{aligned}$$

If $i \neq k$ then it is easy to show that $\operatorname{tr}(P_k dP_i dP_i dP_k) - \operatorname{tr}(P_i dP_k dP_k dP_i) = \operatorname{tr}(dP_i dP_k dP_i)$ and so we finally end up with, restoring the factor $-i/12\pi$,

$$-\frac{i}{4\pi}\sum_{i\neq k}\lambda_i\lambda_k^{-1}\operatorname{tr}(dP_i dP_k dP_k),$$

since $\operatorname{tr}(dP_i dP_k dP_k) = -\operatorname{tr}(dP_k dP_i dP_i)$ if $i \neq k$. Therefore, combining this expression with (B.10) we end up with the following expression for $p^*(2\pi i\nu)$:

$$\begin{aligned}\frac{i}{4\pi}\sum_{i\neq k}\left(\lambda_i^{-1}d\lambda_i - d\lambda_k\lambda_k^{-1} - \lambda_k^{-1}d\lambda_i + \lambda_k^{-1}d\lambda_k\lambda_k^{-1}\lambda_i\right)\operatorname{tr}(P_i dP_k dP_k)\\ -\frac{i}{4\pi}\sum_{i\neq k}\lambda_i\lambda_k^{-1}\operatorname{tr}(dP_i dP_k dP_k).\end{aligned}$$

If we further pull this back to $\Omega^3(G/T\times Y_{T,\mathrm{reg}})$ then we find that it is equal to the expression (B.6) we obtained for $(p_Y)^*df$. This completes the proof of Theorem 5.1.

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